LOGIC

In everyday use, "Go see *Star Wars* tonight" might be considered a statement. But in logic and math, it's not a statement because a ***statement*** is an assertion that can clearly be classified as either *true* or *false* (even if we don't know which one).

Rodin's *The Thinker*

❑ *STATEMENTS AND IMPLICATIONS*

"*The quadratic formula works for all quadratic equations*" is a ***statement*** (and it's true). Another statement is "*2 + 2 = 5*" (but it's false). Even "*There are 100 billion stars in our galaxy*" is a statement. I have no idea whether it's true or false, but I'm sure that it's either true or false.

Most of the results in mathematics are actually a special hooking-together of two statements; we call these ***implications***. An implication is sometimes called an "***if-then***" statement. If p and q are statements, then an implication can be written

If *p*, then *q*

or, ***p* implies *q***

or, $\boldsymbol{p \Rightarrow q}$

The essence of computer programming lies in the *if-then* statement.

The statement $\boldsymbol{p}$ is called the ***hypothesis*** and the statement $\boldsymbol{q}$ is called the ***conclusion***.

For example, the fact that the square of an odd number is odd (for example, $5^2 = 25$) can be written

If n is odd, then n^2 is odd,

or n odd $\Rightarrow$ n^2 odd.

Hypothesis: n odd

Conclusion: n^2 odd

Homework

1. Is the following a statement? "There are exactly 7 people in New York City with 32,000 hairs on their heads."

2. Is the following a statement? "Walk tall!"

3. Is the following a statement? "Arvis said, 'Walk tall!'"

4. Goldbach stated many years ago, "Every **even** number starting at 4 is the sum of two **primes**." For example, 20 is an even number, and **20 = 17 + 3**, where 17 and 3 are prime. As of today, this conjecture has never been proven to be true or false — mathematicians simply don't know. Is Goldbach's Conjecture a statement?

5. State the hypothesis and conclusion of the statement "If x is any real number, then $x^2 \geq 0$."

6. State the Pythagorean Theorem in the "if-then" form.

7. "All dogs go to heaven" is an implication. Write it in the "if-then" form.

8. Consider the implication, "I'll get an A if I study hard enough." State the hypothesis and conclusion.

9. Convert each implication into the

   "If *hypothesis*, then *conclusion*" form:

   a. Whenever I eat pie, I get sick.
   b. The sidewalks get wet whenever it rains.
   c. I get A's if I study.
   d. The President must be at least 35 years of age.

❑ NEGATION

The ***negation*** of a statement is the "opposite" of the statement, and is denoted with a squiggle, ~, in front of the statement, although there are other notations. For example, let p be the statement "T is a right triangle." Then $\sim p$ is read "not p" and is the statement "T is <u>not</u> a right triangle."

A more subtle example: Let q = "All prime numbers are odd." Then $\sim q$ = "Not all prime numbers are odd," meaning, "There is at least one prime number which is not odd." [Can you name it?]

The bottom line for negation is that if p is true, then $\sim p$ is false — and if p is false, then $\sim p$ is true. Can you see that $\sim(\sim p)$ has the same truth value of p? What does the statement "Nobody doesn't like Sara Lee" mean?

p	$\sim p$
True	False
False	True

From the implication $p \Rightarrow q$ and the negation ~, we can create two variations of the implication, described in the following two sections.

Homework

10. Let p be the statement, "The 49ers lost the Super Bowl." Assume that the game cannot end in a tie.
    State $\sim p$.

11. Let p be the statement "$x > 0$". State $\sim p$.

12. Let p be the statement, "Every triangle is a polygon." State $\sim(\sim p)$.

13. Simplify: $\sim(\sim(\sim(\sim p)))$

❑ CONVERSE

The ***converse*** of the statement $p \Rightarrow q$ is $q \Rightarrow p$. We interchange the hypothesis and the conclusion. Consider the statement, "If you live in San Francisco, then you live in California." Its converse is "If you live in California, then you live in San Francisco." The original statement is true, but its converse can easily be false.

What's the converse of the statement "If a triangle has three equal sides, then each of its angles is 60°"? It's the statement, "If each angle of a triangle is 60°, then the triangle has three equal sides." In this example, both the statement and its converse are true.

In short, the converse of a true statement may — or may not — be true.

❑ CONTRAPOSITIVE

Consider the true statement above, "If you live in San Francisco, then you live in California." Though the converse of this statement is false, let's take the converse <u>and</u> negate each part. This is called constructing the ***contrapositive***. If we do this "switch and negate" maneuver, we get the statement, "If you don't live in California, then you don't live in San Francisco." This statement makes sense. Do you agree?

Therefore, the contrapositive of the statement $p \Rightarrow q$ is $\sim q \Rightarrow \sim p$. It seems, at least using the above example, that the contrapositive of a true statement is true, and indeed this is always the case.

Statement	$p \Rightarrow q$	
Its Converse	$q \Rightarrow p$	Truth value has no relation to the truth of the original statement.
Its Contrapositive	$\sim q \Rightarrow \sim p$	Truth value must match that of the original statement.

Homework

14. Let p = "If it's dark, then it's scary."

    a. State the converse of p.

    b. State the contrapositive of p.

    c. Which of the two statements must have the same truth value as p?

15. Let q = "If angle A is a right angle, then A = 90°." State the converse. Is the converse true or false?

16. Let r = "If $x = 5$, then $x^2 = 25$." State the converse. Is the converse true or false?

17. State the converse of the Pythagorean Theorem. Is the converse true or false?

18. Consider the true statement from calculus: "If f is differentiable at x, then f is continuous at x." State the converse and the contrapositive of the statement. Which of the two statements must be true?

19. Consider the false statement from a branch of math called topology: "If $\varnothing$ is in T, then T is a topology." State the converse and the contrapositive of the statement. Which of the two statements must be false?

20. Consider the implication $r \Rightarrow s$. State the converse and the contrapositive of the implication. Which of the two statements must have the same truth value as the original implication?

❑ MORE ON COMBINING STATEMENTS

AND Another way to hook two statements together to make a new statement is to place the word *AND* between them. The meaning is the same as that in English. If both p and q are true, then the statement p AND q is true. But if either p or q (or both) is false, then the statement p AND q is false.

p	q	p AND q
True	True	True
True	False	False
False	True	False
False	False	False

The statement "2 + 2 = 4 AND The letter a is a vowel" is true.
The statement "2 + 2 = 4 AND The letter m is a vowel" is false.
The statement "2 + 2 = 5 AND The letter z is a vowel" is false.

OR Two statements can be linked by the word *OR* to create a new statement. In this case, the logical meaning of OR may not be consistent with the use of the word OR in English. In math, as long as at least one of the statements p or q is true, then the statement p OR q is true. In particular, if both p and q are true, then p OR q is true.

p	q	p OR q
True	True	True
True	False	True
False	True	True
False	False	False

Thus, the following are true statements (since at least one part is true):

2 + 2 = 4 OR Lincoln is our current president.
2 + 2 = 4 OR California is in the U.S.A.

But the following statement is false (since both parts are false):

The earth is cubical OR 10 + 10 = 29.

Note that our use of the word OR is <u>not</u> the way your mother meant the word OR when she said, "You can have ice cream OR a cookie."

The OR we use in logic and math – one or the other or both – is called the **inclusive OR**.

The OR that means one or the other – but <u>not</u> both – is called the **exclusive OR**.

Homework

21. a. What are the conditions for p and q that will make the statement p AND q true?

    b. What will make p AND q false?

22. a. What are the conditions for p and q that will make the statement p OR q true?

    b. What will make p OR q false?

23. T/F: a. "2 + 2 = 4 OR Lincoln was the 1st president."
    b. "2 + 2 = 4 OR $5^2 = 25$.
    c. "1 + 3 = 10 OR I'm 10 feet tall."

24. T/F: a. "2 + 2 = 4 AND Lincoln was the 1st president."
    b. "2 + 2 = 4 AND Washington was the 1st president."
    c. "California is in Idaho AND a circle is a square."

25. "$6 \le 6$" is a true statement. Explain.

26. When is the statement p AND q AND r true?

27. When is the statement p OR q OR r true?

28. [Optional] Consider the logical connective, denoted by *XOR*, and defined by the truth table:

p	q	p XOR q
True	True	False
True	False	True
False	True	True
False	False	False

Give an example from real life that uses this connective.

❑ *Definitions, Axioms, and Theorems*

There are really only two kinds of statements in all of mathematics: those you don't prove (the definitions and axioms), and those you do prove (the theorems).

Consider the following ***definition***: A *right angle* is an angle whose measure is 90°. [The Babylonians adored the number 360, and 90 is one of its many factors.] Is this definition right or wrong? It's not right or wrong — it just is. If you disagree with the definition, then you can either accept it so you'll pass your math course, or you can define it differently and perhaps create a new branch of mathematics!

An example of an ***axiom*** is the *commutative property* for addition of two numbers: $a + b = b + a$. We accept this as fact — we don't consider whether or not there's a proof. Why do we accept this axiom? Well, it seems to work for all the number systems we've seen in algebra — it seems "intuitively" clear. [Not necessarily the case in advanced math courses.]

A ***theorem***, on the other hand, is a statement that requires a **proof**. The proof of a theorem is based on definitions, axioms, and previously proved theorems. The *Quadratic Formula* is an example of a theorem, and can be expressed as an implication:

If $ax^2 + bx + c = 0$ (where $a \neq 0$), **then** $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$.

The classic theorem is the *Pythagorean Theorem*:

If a and b are the legs of a right triangle and c is the hypotenuse, **then** $a^2 + b^2 = c^2$.

At the end of a proof of a theorem we sometimes write Q.E.D., which is Latin for *quod erat demonstrandum*, which loosely translates to "as was to be proved."

Homework

29. A *group* is a semigroup with identity. Is this definition right or wrong?

30. Prove the axiom: If a and b are numbers, then $ab = ba$.

31. The formula for *slope* $(m = \frac{\Delta y}{\Delta x})$ is not a theorem. What is it, then?

32. a. Why is the Distance Formula in the plane considered a theorem?
    b. State the hypothesis and conclusion.
    c. What geometry theorem is the Distance Formula based upon?

33. State the hypothesis and conclusion of the Quadratic Formula.

❑ *EXAMPLES AND COUNTEREXAMPLES*

Consider the statement: If x is any number, then $x^2x^3 = x^5$. Here's an ***example*** of this.

Let $x = 2$. Then,

x^2x^3 gives $2^2 \cdot 2^3 = 4 \cdot 8 = 32$; and

x^5 gives $2^5 = 32$. ✓

What have we proved? Not much — just that $2^2 \cdot 2^3 = 2^5$. But this in no way proves the more general statement, $x^2x^3 = x^5$. It is just an example. It gives some credence to the general statement, but nothing more.

Now consider the statement: If a and b are numbers, then $(a+b)^2 = a^2 + b^2$. Though this statement is true sometimes, it's considered to be a false statement. To prove that a statement is false, all we need is one ***counterexample***: an example that runs counter (contrary) to the statement.

Let $a = 3$ and $b = 4$. Then,

$$(3+4)^2 = 7^2 = 49,$$

whereas $3^2 + 4^2 = 9 + 16 = 25$. ☹

Therefore, $(a+b)^2 = a^2 + b^2$ is a false statement, and it took only one counterexample to prove that it's false.

Homework

34. Consider the statement "$x \ge 0$."
    a. Describe the statement in words.
    b. Give two different x values that make the statement true, and don't make both positive.

35. Consider the statement: $x - y = y - x$ for any real numbers. Is the statement true or false? Prove your answer.

36. Describe precisely when $(a+b)^2 = a^2 + b^2$ is a true statement.

37. A natural number with precisely two factors is called a *prime number*. The first few prime numbers are 2, 3, 5, 7, 11, 13, 17, 19, . . .

    a. T/F: Every prime number is odd. Prove your answer.

    b. T/F: Every odd number is prime. Prove your answer.

38. Disprove the statement: $x^2 = 2x$ for any real number x. Hint: Don't let $x = 0$ or 2. Explain why I'm not allowing you to use 0 or 2.

Review Problems

39. a. Give an example of a true statement.
    b. Give an example of a false statement.
    c. Give an example of a statement whose truth is unknown.
    d. Give an example of a sentence which is not a statement.

40. What distinguishes a statement from a non-statement?

41. A *semigroup* is defined to be a set with an associative binary operation. Prove this statement.

42. Simplify the logic expression $\sim(\sim(\sim p))$.

43. A true theorem from group theory states: If H is a subgroup of G, then the order of H divides the order of G.
    a. State the hypothesis of this theorem.
    b. State the converse and the contrapositive of this theorem.
    c. Which of the two statements in part b <u>must</u> be true?

44. Let p = "2 + 3 = 5" and q = "Lincoln was the 2nd president."
    a. Is p OR q true or false?   b. Is p AND q true or false?

45. a. A "lemma" is a math statement similar to a theorem. Is a lemma a statement we do, or do not, prove?
    b. A "postulate" is a math statement similar to an axiom. Is a postulate a statement we do, or do not, prove?

46. Consider the statement $(x + y)^3 = x^3 + y^3$ for any real numbers x and y.
    a. Give an example to give credibility to the statement.
    b. Now prove that the statement is false. What did you use to prove it's false?

47. Explain why "Ultraviolet waves have a higher frequency than infrared waves" is a *statement* in logic.

48. One of the axioms of a group is that there must be an identity element. Prove this statement.

49. Simplify the logic expression $\sim(\sim(\sim(\sim(\sim p))))$.

50. A true theorem from group theory states: If H is a subgroup of a cyclic group G, then H is cyclic.
    a. State the conclusion of this theorem.
    b. State the converse and the contrapositive of this theorem.
    c. Which of the two statements in part b <u>must</u> be true?

51. Let p = "$2 + 3 = 9$" and q = "Lincoln was the 2nd president."
    a. Is p OR q true or false?
    b. Is p AND q true or false?

52. Consider the statement $\sqrt{a^2 + b^2} = a + b$ for any real numbers a and b.
    a. Give an example to give some credibility to the statement.
    b. Now prove that the statement is false. What did you use to prove it's false?

53. True/False

a. $p \Rightarrow q$ is called an implication.

b. In $p \Rightarrow q$, q is called the hypothesis.

c. $\sim(\sim p)$ simplifies to $\sim p$.

d. The converse of $p \Rightarrow q$ is $\sim p \Rightarrow \sim q$.

e. The contrapositive of $\sim q \Rightarrow \sim p$ is $p \Rightarrow q$.

f. $x = 2$ is a counterexample to the conjecture $x^2 = 2x$.

g. If p is true, q is false, and r is true, then (p OR q) AND r is true.

h. Axioms and theorems are proved.

i. Definitions are never proved.

j. The converse of a true implication is always true.

k. The contrapositive of a false implication is always false.

Solutions

1. Yes — I don't know whether it's true or false, but it's certainly one or the other.

2. No — It's simply a command. There's no possible way the sentence can be evaluated as true or false.

3. Yes, this sentence <u>is</u> a statement. Even though it sounds like the previous problem, it makes the statement that Arvis said something. Either he did or he didn't, so the statement is either true or false.

4. Yes — as we're learning, a statement is a sentence with a truth value of either true or false, regardless of whether we mortals know which it is. By the way, 100 is the sum of what two primes?

5. Hypothesis: If x is any real number — Conclusion: $x^2 \geq 0$.

6. If a and b are the legs of a right triangle and c is the hypotenuse, then

$a^2 + b^2 = c^2$.

7. If you're a dog, then you'll go to heaven.

8. Hypothesis: If I study hard enough — Conclusion: I'll get an A

9. a. If I eat pie, then I get sick.
  b. If it rains, then the sidewalks get wet.
  c. If I study, then I get A's.
  d. If one is the President, then one is at least 35 years of age.

10. The 49ers won the Super Bowl.

11. Hint: The answer is <u>not</u> "$x < 0$."

12. Every triangle is a polygon.

13. p

14. a. If it's scary, then it's dark b. If it's not scary, then it's not dark.
  c. The contrapositive always has the same truth value as the original statement.

15. If A = 90°, then A is a right angle. The converse is true.

16. If $x^2 = 25$, then $x = 5$. The converse is false (since $x = \pm 5$).

17. If a, b, and c are the sides of a triangle and $a^2 + b^2 = c^2$, then the triangle is a right triangle. The converse is true.

18. Converse: If f is continuous at x, then f is differentiable at x.
Contrapositive: If f is not continuous at x, then f is not differentiable at x.
The contrapositive must be true if the original statement is true.

19. Converse: If T is a topology, then $\varnothing$ is in T.
Contrapositive: If T is not a topology, then $\varnothing$ is not in T.
The contrapositive will be false.

20. Converse: $s \Rightarrow r$ Contrapositive: $\sim s \Rightarrow \sim r$
The contrapositive always has the same truth value as the original statement.

21. a. Both p and q must be true.
  b. If either or both of them are false.

22. a. Either p is true, or q is true, or both are true.
  b. Only if both of them are false.

23. a. T b. T c. F

24. a. F b. T d. F

25. The statement says, "6 is less than OR equal to 6." Since 6 is equal to 6, and since an OR statement is true as long as one of the parts is true, the entire statement is true.

26. Only when all three parts are true.

27. As long as any one of them (or two of them or all of them) is true.

28. I'm curious what you think the truth table really means.

29. Neither — it's a definition.

30. Axioms aren't proven — they're accepted.

31. It's a definition. (It's not a theorem because it cannot be proven.)

32. a. Because it can be derived (proved).

b. Hypothesis: If (a, b) and (c, d) are points in the plane . . .

Conclusion: then the distance between them is $\sqrt{(a-c)^2+(b-d)^2}$

c. The Pythagorean Theorem

33. Hypothesis: Given the quadratic equation $ax^2 + bx + c = 0$ . . .

Conclusion: the solutions are $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

34. a. "x is greater than or equal to zero."

b. If $x = 7$, the statement is true, and if $x = 0$, the statement is true.

35. False — to prove this, we take the *counterexample*: $7 - 2 = 5$, whereas $2 - 7 = -5$.

36. It has something to do with zero.

37. a. See the 2? b. Is 15 prime?

38. Let $x = 5$ and the statement fails. Choosing x as 0 or 2 is a lousy choice because the statement actually works for these numbers.

39. a. For example, "The *Harry Potter* books were written by a woman."

b. For example, "Every parabola has two x-intercepts."

c. For example, "It took exactly 197 geeks to write Windows."

d. For example, "Why did you break the window?"

40. A statement can be assigned a true/false value.

41. A definition cannot be proved.

42. ~ p

43. a. H is a subgroup of G.

b. *converse*: If the order of H divides the order of G, then H is a subgroup of G.

contrapositive: If the order of H does not divide the order of G, then H is not a subgroup of G.

c. The contrapositive must be true.

44. a. True b. False **45**. a. Prove b. Do not prove

46. a. Let $x = 1$ and $y = 0$ and the statement is true. b. One counterexample is to let $x = 2$ and $y = 3$.

47. It's a statement because it's an assertion that is either true or false.

48. It can't be proved; it's an axiom, which we assume to be true.

49. ~ p

50. a. H is cyclic.

b. *converse*: If H is cyclic, then H is a subgroup of a cyclic group G.

contrapositive: If H is not cyclic, then H is not a subgroup of a cyclic group G.

c. The contrapositive must be true.

51. a. false b. false

52. a. Letting $a = 5$ and $b = 0$, for instance, will make the statement true.

b. Letting $a = 4$ and $b = 3$ will make the statement false. We used a *counterexample* to prove the statement false.

53. a. T b. F c. F d. F e. T f. F

g. T h. F i. T j. F k. T

"The opposite of a correct statement is a false statement. But the opposite of a profound truth may well be another profound truth."

Niels Bohr, physicist (1885–1962)